Journal of Approximation Theory **92**, 245–266 (1998)
Article No. AT973118

On Power Functions and Error Estimates for Radial Basis Function Interpolation

Will Light* and Henry Wayne†

Department of Mathematics and Computer Science, University of Leicester, Leicester, United Kingdom

Communicated by R. Schaback

Received March 29, 1996

This paper discusses approximation errors for interpolation in a variational setting which may be obtained from the analysis given by Golomb and Weinberger. We show how this analysis may be used to derive the power function estimate of the error as introduced by Schaback and Powell. A simple error tool for the power function is presented, which is similar to one appearing in the work of Madych and Nelson. It is then shown that this tool is adequate to reproducing the original error analysis presented by Duchon. An interesting consequence of our work is that no explicit use is made of the polynomial reproduction properties of the interpolation operator.

1. INTRODUCTION

This paper discusses interpolation of real-valued functions on a set $\Omega \subset \mathbb{R}^n$ by certain rather special subspaces, which include radial basis function interpolation. The set of interpolation points will be $\mathscr{A} = \{a_1, ..., a_m\} \subset \mathbb{R}^n$ and the interpolating subspace at its most elementary will be

$$\operatorname{span}\{\Psi(\cdot - a_i);\, i = 1, ..., m\},$$

where $\Psi: \mathbb{R}^n \to \mathbb{R}$. Interpolants will then have the form

$$u(x) = \sum_{j=1}^{m} \alpha_j \Psi(x - a_j), \qquad x \in \mathbb{R}^n,$$

where $\alpha_j \in \mathbb{R}$, $j = 1, ..., m$. In radial basis function interpolation, Ψ has the particularly simple form $\Psi(x) = \phi(|x|)$, where $|\cdot|$ is the usual Euclidean

* This author was supported by NATO Research Grant 0095/46.
† This author was supported by a studentship from the Engineering and Physical Sciences Research Council.

0021-9045/98 $25.00

norm on $\mathbb{R}^n$ and $\phi \in C[0, \infty)$. In many cases, it is helpful to have additional polynomial terms in the interpolant. Let Π_{k-1} denote the subspace of $C(\mathbb{R}^n)$ consisting of polynomials of (total) degree at most $k-1$. Let $\dim \Pi_{k-1} = \ell$ and let $p_1, ..., p_\ell$ be a basis for Π_{k-1}. Then the interpolant has the form

$$u(x) = \sum_{j=1}^{m} \alpha_j \Psi(x - a_j) + \sum_{i=1}^{\ell} \beta_i p_i(x), \qquad x \in \mathbb{R}^n. \tag{1}$$

There are so-called "natural" conditions which, when added to the interpolation conditions, specify the interpolant uniquely in many cases. Suppose data $d_1, ..., d_m$ is prescribed on $a_1, ..., a_m$. Then the requirements on the interpolant are $u(a_j) = d_j$, $j = 1, ..., m$ (the interpolation conditions) and $\sum_{j=1}^{m} \alpha_j p_i(a_j) = 0$, $i = 1, ..., \ell$ (the "natural" conditions). One can write this system in matrix form as

$$\begin{pmatrix} A & P \\ P^T & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} d \\ 0 \end{pmatrix},$$

where A is an $m \times m$ matrix with (i, j) element $\Psi(a_i - a_j)$, and P is an $m \times \ell$ matrix with (i, j) element $p_j(a_i)$. The vectors α, β and d have the obvious definitions. Two conditions should hold for this system to have a unique solution for all values of the data d. Firstly, the matrix A should be nonsingular over the subspace of vectors α satisfying $P^T\alpha = 0$. Secondly, polynomials in Π_{k-1} should be uniquely determined by their values on $\mathscr{A}$, that is, if $p \in \Pi_{k-1}$ and $p(a_i) = 0$, $i = 1, ..., m$, then $p = 0$. In this case $\mathscr{A}$ is said to be unisolvent with respect to Π_{k-1}.

A central principle in such interpolation problems is that as the set $\mathscr{A}$ "fills out" Ω, the error between a function and its interpolant should go to zero. The usual measure for the way $\mathscr{A}$ "fills out" Ω is $h = \sup_{t \in \bar{\Omega}} \inf_{a \in \mathscr{A}} |t - a|$. If $f \in C(\Omega)$ say, and Uf is its interpolant, then one might hope to get $\|f - Uf\| = \mathcal{O}(h^\lambda)$ as $h \to 0$, where λ is some measure of the smoothness of f. We will establish such error bounds in this paper.

Early work in this area was due to Duchon [3], who developed the theory of surface splines. His error estimates rest crucially on the property that his interpolant preserves polynomials of degree $k-1$. (Since his interpolants are all special cases of the ones just described, the polynomial preservation property of U, that is, $Up = p$ for all $p \in \Pi_{k-1}$, is clear.) Another approach, taken by Madych and Nelson [7]. Powell [10] and Schaback [12] uses a pointwise error estimator. This estimator involves an expression which Schaback calls the power function. Both Powell and Schaback compute this power function in some sense. Their estimates coincide with those of Duchon, although one should note that Schaback is interested in a much wider range of examples. Also, Powell is very careful in his computation of the constants involved in the order estimates, and gets the best known constants for thin-plate splines.

Our purpose in this paper is to work in the setting of both Duchon and Madych and Nelson, using the power function as the tool for error analysis. The main effect of this approach is that the polynomial reproduction properties of the interpolant make no overt appearance in our proofs. Section two shows (for the first time, we believe) the precise connection between the variational theory of Golomb and Weinberger and the power function of Schaback. Section three presents a short, analysis of the power function when used as an error estimate. Section four shows how to obtain the error estimates of Duchon using these techniques.

We conclude this section with two examples of our results. Let Ω be a subset of $\mathbb{R}^n$ which is open, bounded, connected and has the cone property. For each $h>0$, let $\mathscr{A}_h$ be a finite, Π_{k-1}-unisolvent subset of Ω with $\sup_{t\in\bar{\Omega}} \inf_{a\in\mathscr{A}_h} |t-a|<h$. Let j be the smallest integer greater than or equal to $1+(n/2)$. Let $f\in C^{(j)}(\bar{\Omega})$, and let $U_h f$ denote the interpolant specified in Eq. (1), satisfying $(Uf)(a)=f(a)$ for all $a\in\mathscr{A}_h$. If $\Psi(r)=r^2 \ln r$ and $\ell=n+1$ in Eq. (1), then a consequence of 4.2 is that $|f(x)-(U_h f)(x)|=\mathscr{O}(h)$ as $h\to 0$, for all $x\in\Omega$. If $\Psi(r)=r^3$ and $\ell=n+1$ in Eq. (1), then we need to take j to be the smallest integer greater than or equal to $(n+3)/2$. Let $f\in C^{(j)}(\bar{\Omega})$. Then $|f(x)-(U_h f)(x)|=\mathscr{O}(h^{3/2})$ as $h\to 0$ for all $x\in\Omega$. We emphasize that in contrast to other authors (Meinguet [9], Powell [10]) we do not make any assumption about the dimension n here. Also the assumption that $\ell=n+1$ is supposed to convey to the reader that linear polynomials are being used in both these interpolants.

2. VARIATIONAL THEORY

In this short section we describe how the salient features of the seminal paper by Golomb and Weinberger apply to our situation. Let X be a linear space of continuous, real-valued functions on $\mathbb{R}^n$. Let $\langle\cdot,\cdot\rangle : X\times X\to\mathbb{R}$ be a semi-inner product on X with finite dimensional kernel K having dimension ℓ. We will assume $\mathscr{A}=\{a_1, ..., a_m\}$ is a set of points in $\mathbb{R}^n$ which is unisolvent with respect to K. That is, if $p\in K$ and $p(a_i)=0$, $i=1, ..., \ell$, then $p=0$. We will also assume that given $x\in\mathbb{R}^n$, there exists $M>0$ such that $|g(x)|\leqslant M\langle g, g\rangle$ for all $g\in X$ such that $g(a_i)=0$, $i=1, ..., m$. We may now form an inner product on X,

$$(u, v)=\langle u, v\rangle+\sum_{i=1}^{\ell} u(a_i)\, v(a_i), \qquad u, v\in X. \tag{2}$$

Because of our previous hypothesis, point evaluations are continuous linear functionals on X when X derives its topology from the norm induced by this inner product. We will set $\|u\| = \sqrt{(u, u)}$ and assume X is complete with respect to this norm.

Fix $x \in \mathbb{R}^n$ such that the point evaluation functionals at $x, a_1, ..., a_m$ are linearly independent over X. By the Riesz representation theorem, there exist $q, q_1, ..., q_m \in X$ such that $u(x) = (u, q)$ and $u(a_i) = (u, q_i)$, $i = 1, ..., m$ for all $u \in X$. Define the set G by

$$G = \{ g \in X\colon g|_{\mathscr{A}} = 0 \},$$

and let w be the element of G such that $\|w\| = 1$ and

$$w(x) = \sup\{ |g(x)|\colon g \in G,\ \|g\| = 1 \}.$$

Note that $(G, \|\cdot\|)$ is again a Hilbert space, and so the Riesz representation theorem guarantees the uniqueness of w. Let $\|\cdot\|$ be the norm induced by the inner product. Fix $f \in X$. The closed, convex set

$$\{u \in X\colon u|_{\mathscr{A}} = f|_{\mathscr{A}}\},$$

has a unique point of minimal norm. We denote this element by Uf and refer to it as the *minimal norm interpolant to f*. It is this interpolant which we will concentrate on in our analysis. Because $f - Uf \in G$ and Uf is perpendicular to G,

$$\begin{aligned} |f(x) - (Uf)(x)|^2 &\leqslant w(x)^2\ \|f - Uf\|^2 \\ &= w(x)^2\ \langle f - Uf, f - Uf \rangle \\ &= w(x)^2\ \{\langle f, f \rangle - \langle Uf, Uf \rangle\} \\ &\leqslant w(x)^2\ \langle f, f \rangle. \end{aligned} \tag{3}$$

The element $w(x)$ is essentially the power function of Schaback, although it is not easily recognised as such in this abstract form. Also, the bound given in (3) is best possible, since one may easily check that $|w(x) - (Uw)(x)| = w(x) \sqrt{\langle w, w \rangle}$. Let $P\colon X \to G$ denote the orthogonal projection. Then

$$\begin{aligned} w(x) &= \sup\{ |g(x)|\colon g \in G,\ \|g\| = 1 \} \\ &= \sup\{ |(g, Pq)|\colon g \in G,\ \|g\| = 1 \} \\ &= \|Pq\| \\ &= \sqrt{\langle Pq, Pq \rangle}. \end{aligned}$$

Unfortunately, the above observation doesn't really help to compute $w(x)$.

2.1. Lemma. *Let* $a_1, ..., a_m \in \mathbb{R}^n$ *have representers* $q_1, ..., q_m \in X$, *so that* $f(a_i) = (f, q_i)$, $i = 1, ..., m$, *for all* $f \in X$. *Let* $x \in \mathbb{R}^n$ *have representer* q_0. *Then there exist* $\alpha_0, ..., \alpha_m \in \mathbb{R}$ *such that* $w = \sum_{i=0}^m \alpha_i q_i$. *Furthermore, these coefficients are determined by the equations* $\|w\| = 1$ *and*

$$0 = w(a_j) = \sum_{i=0}^m \alpha_i (q_i, q_j), \; j = 1, ..., m.$$

Proof. Define

$$G_x = \{ g \in X \colon g|_{\mathcal{A}} = 0 \text{ and } g(x) = 0 \}.$$

Using the representers, we can write

$$G_x = \{ g \in X \colon (g, q_i) = 0, \; i = 0, ..., m \}. \tag{4}$$

Now let $Q \colon X \to G_x$ be the orthogonal projection. Since $w|_{\mathcal{A}} = 0$, $(w - Qw)|_{\mathcal{A}} = 0$. Hence $w - Qw \in G$. Also, $w(x) - (Qw)(x) = w(x)$. Finally, $\|w - Qw\| \leqslant \|w\| = 1$. By the uniqueness of w, it now follows that $Qw = 0$. Thus $w \in G_x^\perp$. From (4) it follows that there exist $\alpha_0, ..., \alpha_m \in \mathbb{R}$ such that $w = \sum_{i=0}^m \alpha_i q_i$. The coefficients $\alpha_0, ..., \alpha_m$ are defined by the equations $\|w\| = 1$ and

$$\sum_{i=0}^m \alpha_i (q_i, q_j) = \sum_{i=0}^m \alpha_i q_i(a_j) = w(a_j) = 0, \qquad j = 1, ..., m. \quad \blacksquare$$

2.2. Lemma. *Let* $p_1, ..., p_\ell \in K$ *be such that* $p_i(a_j) = \delta_{ij}$, $i, j = 1, ..., \ell$. *Then for all* $f \in X$, $f(a_i) = (f, p_i)$, $i = 1, ..., \ell$.

Proof. The conclusion of the Lemma follows immediately from

$$(f, p_i) = \langle f, p_i \rangle + \sum_{j=1}^{\ell} f(a_j)\, p_j(a_i) = 0 + \sum_{j=1}^{\ell} f(a_j)\, p_j(a_i) = f(a_i). \quad \blacksquare$$

2.3. Lemma. *Let* $\alpha_0, ..., \alpha_m$ *be defined as in* 2.1, *and* $p_1, ..., p_\ell$ *as in* 2.2. *Then* $\sum_{r=0}^m \alpha_r p(a_r) = 0$ *for all* $p \in K$.

Proof. We have, by 2.1, for $i = 1, ..., \ell$,

$$\sum_{r=0}^m \alpha_r p_i(a_r) = \sum_{r=0}^m \alpha_r (p_i, q_r) = \left(p_i, \sum_{r=0}^m \alpha_r q_r \right) = \sum_{r=0}^m \alpha_r q_r(a_i) = 0.$$

Since $p_1, ..., p_\ell$ is a basis for K, the result follows. $\blacksquare$

We now make explicit our assumptions about the sort of spaces we are considering.

2.4. Assumption. We will suppose $X \subset C(\mathbb{R}^n)$ and the following hold:

(i) A semi-inner product $\langle \cdot, \cdot \rangle$ is defined on X with kernel K,

(ii) Let $a_1, ..., a_\ell$ be unisolvent with respect to K and let $p_1, ..., p_\ell \in K$ satisfy $p_i(a_j) = \delta_{ij}$, $1 \leqslant i, j \leqslant \ell$. Suppose $\phi \in C(\mathbb{R})$ is such that, for each $x \in \mathbb{R}^n$,

$$r_x(y) = \phi(|y - x|) - \sum_{i=1}^{\ell} p_i(x)\, \phi(|y - a_i|), \; y \in \mathbb{R}^n,$$

defines a function $r_x \in X$ with $(f, r_x) = f(x)$ for all $f \in X$ such that $f(a_1) = \cdots = f(a_\ell) = 0$. Here, for $u, v \in X$,

$$(u, v) = \sum_{i=1}^{\ell} u(a_i)\, v(a_i) + \langle u, v \rangle.$$

The above assumption describes the representer for the point evaluation at x, at least for a subset of functions in X. Although the form of r_x may look overly elaborate, there are two key principles here. Firstly, if $f \in X$ and $f(a_1) = \cdots = f(a_\ell) = 0$, then $(f, r_x) = \langle f, r_x \rangle$. In fact, $\langle f, \phi(|\cdot - x|) \rangle$ is usually well-defined for f in some subset of X. For example, in the cases considered by Duchon, f should be a compactly supported, infinitely differentiable function. However, $\phi(|\cdot|)$ is not usually itself a member of X. One has to take linear combinations of the form $\sum_{i=0}^{m} \lambda_i \phi(|\cdot - b_i|)$, and demand that $\sum_{i=0}^{m} \lambda_i p(b_i) = 0$ for all $p \in K$ in order that this linear combination is an element of X. Looking back to the form of r_x, we see that the coefficients satisfy

$$p(x) - \sum_{i=1}^{\ell} p_i(x)\, p(a_i) = p(x) - p(x) = 0,$$

as required.

2.5. Lemma. *Suppose X satisfies 2.4. Then the representer of the point evaluation at $x \in \mathbb{R}^n$ (i.e., the element $q_x \in X$ such that $(f, q_x) = f(x)$ for all $f \in X$) has the form*

$$q_x(y) = \phi(|y - x|) - \sum_{i=1}^{\ell} p_i(x)\, \phi(|y - a_i|) - \sum_{i=1}^{\ell} p_i(y)\, \phi(|x - a_i|)$$
$$+ \sum_{i,j=1}^{\ell} \phi(|a_i - a_j|)\, p_j(y)\, p_i(x) + \sum_{i=1}^{\ell} p_i(y)\, p_i(x).$$

Proof. Firstly, let $\Gamma = \{u \in X\colon u(a_1) = \cdots = u(a_\ell) = 0\}$. Then define $P\colon X \to X$ by $Pf = \sum_{i=1}^{\ell} f(a_i)\, p_i$. It is easy to check that P is the orthogonal projection of X onto $\Gamma^\perp$, and so $I - P$ is the orthogonal projection onto Γ. Now,

$$(f, (I-P)\, r_x) = \begin{cases} f(x), & f \in \Gamma \\ 0, & f \in \Gamma^\perp \end{cases}.$$

Note from the given formulae for P, r_x and q_x that

$$(I-P)\, r_x = q_x - \sum_{r=1}^{\ell} p_i(x)\, p_i.$$

Now, for any $f \in X$, using 2.2,

$$\begin{aligned} f(x) &= (f - Pf)(x) + (Pf)(x) \\ &= (f - Pf, (I-P)\, r_x) + (Pf)(x) \\ &= (f, (I-P)\, r_x) + \sum_{i=1}^{\ell} (f, p_i)\, p_i(x) \\ &= \left(f, (I-P)\, r_x + \sum_{i=1}^{\ell} p_i(x)\, p_i \right) \\ &= (f, q_x), \end{aligned}$$

as required. ∎

The next theorem is the main one in this section, and explains the connection between the power function of Schaback and Powell and the work of Golomb and Weinberger.

2.6. THEOREM. *Define $a_0 = x$, and suppose $a_0, ..., a_m$ have representers $q_0, ..., q_m$, respectively. Let $\alpha_0, ..., \alpha_m \in \mathbb{R}$ be such that the power function $w = \sum_{i=0}^{m} \alpha_i q_i$. Then,*

$$w(x)^2 = \sum_{r, s=0}^{m} \beta_r \beta_s \phi(|a_r - a_s|),$$

where $\beta_r = \alpha_r / \alpha_0$, $r = 0, ..., m$.

Proof. Set $\psi_z(y) = \phi(|y-z|)$, $y, z, \in \mathbb{R}^n$. Then from 2.5,

$$\begin{aligned} w &= \sum_{r=0}^{m} \alpha_r q_r \\ &= \sum_{r=0}^{m} \alpha_r \left\{ \psi_{a_r} - \sum_{i=1}^{\ell} p_i(a_r)\, \psi_{a_i} - \sum_{i=1}^{\ell} p_i \psi_{a_i}(a_r) \right. \\ &\qquad \left. + \sum_{i,\, j=1}^{\ell} \psi_{a_i}(a_j)\, p_j(a_r)\, p_i + \sum_{i=1}^{\ell} p_i(a_r)\, p_i \right\} \\ &= \sum_{r=0}^{m} \mu_r \psi_{a_r} + \rho, \end{aligned}$$

where ρ belongs to K. We have

$$\mu_r = \begin{cases} \alpha_r, & r = 0, l+1, ..., m \\ \alpha_r - \sum_{s=0}^{m} \alpha_s p_r(a_s), & r = 1, ..., \ell \end{cases},$$

and

$$\rho = \sum_{r=0}^{m} \alpha_r \left\{ \sum_{i,\, j=1}^{\ell} \psi_{a_i}(a_j)\, p_j(a_r)\, p_i - \sum_{i=1}^{\ell} p_i \psi_{a_i}(a_r) + \sum_{i=1}^{\ell} p_i(a_r)\, p_i \right\}.$$

Now from 2.3, $\sum_{s=0}^{m} \alpha_s p_r(a_s) = 0$, $r = 1, ..., \ell$, and so $\mu_r = \alpha_r$, $r = 0, ..., m$. Then, again using 2.3 twice,

$$\begin{aligned} \rho &= -\sum_{i=1}^{\ell} p_i \sum_{r=0}^{m} \alpha_r \psi_{a_i}(a_r) \\ &\quad + \sum_{i,\, j=1}^{\ell} \psi_{a_j}(a_i)\, p_i \sum_{r=0}^{m} \alpha_r p_j(a_r) + \sum_{i=1}^{\ell} p_i \sum_{r=0}^{m} \alpha_r p_i(a_r) \\ &= -\sum_{i=1}^{\ell} p_i \sum_{r=0}^{m} \alpha_r \psi_{a_i}(a_r). \end{aligned}$$

Thus,

$$w = \sum_{r=0}^{m} \alpha_r \psi_{a_r} - \sum_{i=1}^{\ell} p_i \sum_{r=0}^{m} \alpha_r \psi_{a_i}(a_r). \tag{5}$$

Now recall that $w \in G$, so that

$$1 = \|w\|^2 = (w, w) = \left(\sum_{s=0}^{m} \alpha_s q_s, w \right) = \alpha_0 (q_0, w) = \alpha_0 w(x).$$

Also, from (5), with a final application of 2.3,

$$
\begin{aligned}
(w, w) &= \left(\sum_{s=0}^{m} \alpha_s q_s, w \right) \\
&= \sum_{s=0}^{m} \alpha_s w(a_s) \\
&= \sum_{r,s=0}^{m} \alpha_r \alpha_s \psi_{a_r}(a_s) - \sum_{i=1}^{\ell} \sum_{r=0}^{m} \alpha_r \psi_{a_i}(a_r) \sum_{s=0}^{m} \alpha_s p_i(a_s) \\
&= \sum_{r,s=0}^{m} \alpha_r \alpha_s \psi_{a_r}(a_s).
\end{aligned}
$$

Finally,

$$
\begin{aligned}
\sum_{r,s=0}^{m} \beta_r \beta_s \psi_{a_r}(a_s) &= \sum_{r,s=0}^{m} \frac{\alpha_r \alpha_s}{\alpha_0^2} \psi_{a_r}(a_s) \\
&= \frac{1}{\alpha_0^2} \alpha_0 w(x) \\
&= \frac{1}{\alpha_0} w(x) \\
&= w(x)^2. \quad \blacksquare
\end{aligned}
$$

Theorem 2.6 is the power function of Schaback, and has virtually identical form to that of Powell [10]. To deduce the form given in Powell, one simply defines $\gamma_i = -\beta_i$, $i = 1, ..., m$. Then

$$
w(x)^2 = \sum_{r,s=1}^{m} \gamma_r \gamma_s \phi(|a_r - a_s|) - 2 \sum_{r=1}^{m} \gamma_r \phi(|x - a_r|) + \phi(0).
$$

If we now recall that Powell treats a special case where $\phi(0) = 0$, then the right hand side of the above is exactly that obtained by Powell.

2.7. Lemma. *Let A and B be subsets of $\mathbb{R}^n$ which are unisolvent with respect to K. Let the power functions associated with A and B be w_A and w_B respectively. If $A \subset B$, then $w_A \geqslant w_B$.*

Proof. Fix $x \in \mathbb{R}^n$. Define $G_A = \{g \in X: g|_A = 0\}$ and $G_B = \{g \in X: g|_B = 0\}$. Then $G_B \subset G_A$, and so

$$
\begin{aligned}
w_B(x) &= \sup\{|g(x)|: g \in G_B, \|g\| = 1\} \\
&\leqslant \sup\{|g(x)|: g \in G_A, \|g\| = 1\} \\
&= w_A(x). \quad \blacksquare
\end{aligned}
$$

If one is interested in the asymptotic behaviour of the error, which will be the main concern in the next section, then 2.7 can be used to good effect. The next result shows how this is done.

2.8. THEOREM. *Let X satisfy 2.4. Let Uf be the minimal norm interpolant to f on $a_1, ..., a_m \in \mathbb{R}^n$. Suppose $\ell \leqslant m$ and $\{a_1, ..., a_\ell\}$ is a unisolvent set of points with respect to K. Then*

$$|f(x)-(Uf)(x)|^2 \leqslant \left\{\phi(0) - 2\sum_{i=1}^{\ell} p_r(x)\,\phi(|x-a_r|) + \sum_{r,s=1}^{\ell} p_r(x)\,p_s(x)\,\phi(|a_r-a_s|)\right\}\langle f, f\rangle.$$

Proof. From (3),

$$\{f(x)-(Uf)(x)\}^2 \leqslant w(x)^2\,\langle f, f\rangle.$$

Here, of course, w refers to the power function based on the points $a_1, ..., a_m$. However, because of 2.7, the inequality will continue to hold if we regard w as being the power function based on $a_1, ..., a_\ell$. From 2.6,

$$w(x)^2 = \sum_{r,s=0}^{\ell} \beta_r\beta_s\phi(|a_r-a_s|),$$

where $\beta_r = \alpha_r/\alpha_0$ and $w = \sum_{i=0}^{\ell} \alpha_i q_i$. By virtue of 2.2, $w = \alpha_0 q_0 + \sum_{i=1}^{\ell} \alpha_i p_i$. Now, for $j = 1, ..., \ell$,

$$\begin{aligned} 0 = w(a_j) &= \alpha_0 q_0 + \sum_{i=1}^{\ell} \alpha_i p_i(a_j) \\ &= \alpha_0 q_0(a_j) + \alpha_j \\ &= \alpha_0 (q_0, p_j) + \alpha_j \\ &= \alpha_0 p_j(x) + \alpha_j. \end{aligned}$$

Hence, $\alpha_j/\alpha_0 = -p_j(x)$, and the result follows. ∎

The following is a rephrasing of 2.8 which will prove useful.

2.9. COROLLARY. *Let X satisfy 2.4. Suppose $g \in X$ satisfies $g(a_1) = \cdots = g(a_\ell) = 0$, where $\{a_1, ..., a_\ell\}$ is a unisolvent set of points with respect to Π_{k-1}. Then*

$$\{g(x)\}^2 \leqslant \left\{ \phi(0) - 2 \sum_{r=1}^{\ell} p_r(x)\, \phi(|x-a_r|) + \sum_{r,s=1}^{\ell} p_r(x)\, p_s(x)\, \phi(|a_r - a_s|) \right\} \langle g, g \rangle.$$

Proof. One simply notices from (3) that if $g(a_1) = \cdots = g(a_\ell) = 0$, then

$$\{g(x)\}^2 \leqslant \{w(x)\}^2 \langle g, g \rangle.$$

Here, w is the power function based on $a_1, ..., a_\ell$. One then follows 2.8 identically. ▮

3. ERROR ESTIMATES

In this section, we examine the asymptotic behaviour of the power function

$$\phi(0) - 2 \sum_{r=1}^{\ell} p_r(x)\, \phi(|x-a_r|) + \sum_{r,s=1}^{\ell} p_r(x)\, p_s(x)\, \phi(|a_r - a_s|),$$

in the special case that $K = \Pi_{k-1}$. We need some straightforward results from Lagrange interpolation theory. These may be found in Duchon [3], and in a wide variety of other places, particularly in the researches of finite element theory.

3.1. Definition. Let $b = \{b_1, ..., b_\ell\}$ be a set of points in $\mathbb{R}^n$ which is unisolvent with respect to Π_{k-1}. Then $L_b\colon C(\mathbb{R}^n) \to \Pi_{k-1}$ will denote the Lagrange interpolation operator defined by $(L_b f)(b_i) = f(b_i)$, $i = 1, ..., \ell$.

3.2. Lemma. *Let Ω be a closed, bounded subset of $\mathbb{R}^n$, and let $b_1, ..., b_\ell \in \Omega$ be a unisolvent set of points with respect to Π_{k-1}. Let $p_1, ..., p_\ell \in \Pi_{k-1}$ be the cardinal functions defined by $p_i(b_j) = \delta_{ij}$, $i, j = 1, ..., \ell$. Suppose $L_b\colon (C(\Omega), \|\cdot\|_\infty) \to (\Pi_{k-1}, \|\cdot\|_\infty)$. Then*

$$\|L_b\| = \max_{x \in \Omega} \sum_{i=1}^{\ell} |p_i(x)|.$$

3.3. Lemma. *Let $B(x, r) = \{y \in \mathbb{R}^n\colon \|y - x\| \leqslant r\}$. Let $b = (b_1, ..., b_\ell)$ denote an ℓ-tuple of points in $\mathbb{R}^n$ which is unisolvent with respect to Π_{k-1}. Then there exists $\delta > 0$ such that if*

$$c = (c_1, ..., c_\ell) \in B(b_1, \delta) \times \cdots \times B(b_\ell, \delta),$$

then $(c_1, ..., c_\ell)$ *is also a unisolvent set of points. Furthermore, if* Ω *is a closed, bounded, connected set containing* $B(b_i, \delta)$, $i=1, ..., \ell$, *then there exists a constant* $K=K(k, \Omega)$ *such that* $\|L_c\| \leqslant K$ *for all* $c=(c_1, ..., c_\ell) \in B(b_1, \delta) \times \cdots \times B(b_\ell, \delta)$.

Proof. Let $\mathscr{U} = \{(a_1, ..., a_\ell): a_1, ..., a_\ell \in \mathbb{R}^n$ and this set of points is unisolvent with respect to $\Pi_{k-1}\}$. Then $\mathscr{U}$ is an open subset of $(\mathbb{R}^n)^\ell$ (its complement describes an algebraic surface in $(\mathbb{R}^n)^\ell$). This establishes the first part of the lemma. Next, for each $f \in C(\Omega)$, the mapping $c \mapsto \|L_c f\|$ is a continuous mapping from $B(b_1, \delta) \times \cdots \times B(b_\ell, \delta)$ into $\mathbb{R}$. Since the domain of this mapping is compact,

$$\sup\{\|L_c f\| : c=(c_1, ..., c_\ell) \in B(b_1, \delta) \times \cdots \times B(b_\ell, \delta)\} < \infty.$$

The required result now follows from the uniform boundedness principle [11, pp. 44–45]. ▮

3.4. LEMMA. *Let* Ω *be a closed, bounded subset of* $\mathbb{R}^n$ *and suppose* $L_b: (C(\Omega), \|\cdot\|_\infty) \to (\Pi_{k-1}, \|\cdot\|_\infty)$ *is a Lagrange interpolation operator based on the set of points* $b=\{b_1, ..., b_\ell\} \in \Omega$. *Let* σ *be a dilation operator, so that* $\sigma(y)=hy$, $h \in \mathbb{R}_+$, $y \in \mathbb{R}^n$. *Then the operator* $L_{\sigma(b)}: C(\sigma(Q)) \to \Pi_{k-1}$ *has* $\|L_{\sigma(b)}\| = \|L_b\|$.

Proof. Suppose $p_i^b(b_j)=\delta_{ij}$, $i, j=1, ..., \ell$. Then, $(p_i^b \circ \sigma^{-1})(\sigma(b_j))=\delta_{ij}$, $i, j=1, ..., \ell$. Since $p_i^b \circ \sigma^{-1} \in \Pi_{k-1}$, these functions must be the cardinal functions for $L_{\sigma(b)}$. Now

$$\begin{aligned}
\|L_{\sigma(b)}\| &= \max\left\{\sum_{i=1}^{\ell} |(p_i^b \circ \sigma^{-1})(x)| : x \in \sigma(\Omega)\right\} \\
&= \max\left\{\sum_{i=1}^{\ell} |(p_i^b \circ \sigma^{-1})(\sigma(y))| : y \in \Omega\right\} \\
&= \max\left\{\sum_{i=1}^{\ell} |(p_i^b(y)|;\ y \in \Omega\right\} \\
&= \|L_b\|. \quad ▮
\end{aligned}$$

3.5. LEMMA (Duchon). *Let* Ω *be an open subset of* $\mathbb{R}^n$ *having the cone property. Then there exist* M, M_1 *and* $h_0 > 0$ *such that to each* $0 < h < h_0$, *there corresponds a set* $T_h \subset \Omega$ *such that*

(i) $B(t, h) \subset \Omega$ *for all* $t \in T_h$,

(ii) $\Omega \subset \bigcup_{t \in T_h} B(t, Mh)$,

(iii) $\sum_{t \in T_h} \chi_{B(t, Mh)} \leqslant M_1$.

Here χ_A *is the function which has value one on* A *and zero elsewhere.*

The next theorem is similar to one given by Madych and Nelson [8].

3.6. THEOREM. *Let $X \subset C(\mathbb{R}^n)$ satisfy 2.4, with $K = \Pi_{k-1}$. Let Ω be an open, connected subset of $\mathbb{R}^n$ having the cone property. Let $\mathscr{A}$ be a finite, Π_{k-1}-unisolvent subset of Ω and define $h = \sup_{t \in \bar{\Omega}} \inf_{a \in \mathscr{A}} |t-a|$. There exists $h_0 > 0$ and constants $c_1, c_2 > 0$, both independent of h and ϕ, such that*

$$|f(x) - (Uf)(x)|^2 \leqslant c_1 \max_{0 \leqslant r \leqslant c_2 h} |\phi(r) - \phi(0)| \langle f, f \rangle,$$

for all $x \in \Omega$, $f \in X$, and $h < h_0$.

Proof. We begin by taking $v_1, ..., v_\ell$ as a set of Π_{k-1}-unisolvent points in $\mathbb{R}^n$. By 3.3, there exists $\delta > 0$ such that the set $B(v_1, \delta) \times \cdots \times B(v_\ell, \delta)$ is a set of unisolvent points in $(\mathbb{R}^n)^\ell$. Dilation by a factor $1/\delta$ generates a new set of points $u_1, ..., u_\ell$ such that the set $B(u_1, 1) \times \cdots \times B(u_\ell, 1)$ is a Π_{k-1}-unisolvent subset of $(\mathbb{R}^n)^\ell$. Choose $R > 0$ such that $B(u_i, 1) \subset B(0, R)$, $i = 1, ..., \ell$.

Choose h_0 in accordance with 3.5. Fix $h > 0$ with $Rh < h_0$. By 3.5, there exists a set $T_h \subset \Omega$ such that $B(t, Rh) \subset \Omega$ for all $t \in T_h$ and $\bigcup_{t \in T_h} B(t, MRh) \supset \Omega$. Now suppose $x \in \Omega$. Then $x \in B(t, MRh)$ for some $t \in T_h$. Define σ: $B(t, MRh) \to B(0, MR)$ by $\sigma(y) = h^{-1}(y - t)$, $y \in B(t, MRh)$. Each ball $B(u_i, 1)$ must contain at least one image under σ of a point in $\mathscr{A}$. Hence we can select $a_1, ..., a_\ell \in B(t, Rh)$ such that $\sigma(a_i) \in B(u_i, 1)$, $i = 1, ..., \ell$. Let L_a: $C(B(t, MRh)) \to \Pi_{k-1}$ be the Lagrange interpolation operator associated with $a = \{a_1, ..., a_\ell\}$. By 3.4, $\|L_a\| = \|L_{\sigma(a)}\|$. But, by 3.3, there exists a constant K such that $\|L_{\sigma(a)}\| < K$ independent of the particular selection of $a_1, ..., a_\ell$. Now, apply 2.8 for $x \in B(t, MRh)$. Then

$$\begin{aligned}
&\{f(x) - (Uf)(x)\}^2 \\
&\quad \leqslant \left\{\phi(0) - 2 \sum_{i=1}^{\ell} p_r(x)\, \phi(|x - a_r|) + \sum_{r,s=1}^{\ell} p_r(x)\, p_s(x)\, \phi(|a_r - a_s|)\right\} \langle f, f \rangle \\
&\quad = \left\{-2 \sum_{r=1}^{\ell} p_r(x)[\phi(|x - a_r|) - \phi(0)]\right. \\
&\qquad \left. + \sum_{r,s=1}^{\ell} p_r(x)\, p_s(x)[\phi(|a_r - a_s|) - \phi(0)]\right\} \langle f, f \rangle \\
&\quad \leqslant \max_{0 \leqslant |y| \leqslant 2MRh} \{|\phi(|y|) - \phi(0)|\}(2K + K^2)\langle f, f \rangle.
\end{aligned}$$

Setting $c_1 = 2K + K^2$ and $c_2 = 2MR$ gives the required result. ∎

4. SURFACE SPLINES

We now turn to a deeper application of the power function—the surface splines of Duchon. Define a measure μ on $\mathbb{R}^n$ by $d\mu(x)=|x|^{2s}\,dx$, where dx is the usual Lebesgue measure on $\mathbb{R}^n$, and $s\in\mathbb{R}$. We will consider the space

$$X=\{f\colon f \text{ is a distribution with } \widehat{D^\alpha f}\in L^2(\mu),\ \alpha\in\mathbb{Z}^n_+,\ |\alpha|=k\}.$$

We will assume throughout this section that $2k+2s>n$. This has the effect (Duchon [4]) that $X\subset C(\mathbb{R}^n)$. The space X is equipped with the semi-inner product

$$\langle f, g\rangle=\sum_{|\alpha|=k} c_\alpha \int_{\mathbb{R}^n} (\widehat{D^\alpha f})(\overline{\widehat{D^\alpha g}})\,d\mu(x),$$

where $\sum_{|\alpha|=k} c_\alpha \xi^{2\alpha}=|\xi|^{2k}$. The kernel of the semi-inner product is Π_{k-1}, and if $\{a_1, ..., a_\ell\}$ is a Π_{k-1}-unisolvent set of points in $\mathbb{R}^n$ then,

$$(f, g)=\sum_{i=1}^{\ell} f(a_i)\,g(a_i)+\langle f, g\rangle,$$

defines an inner product on X. This inner product induces a norm on X in the usual way. When X is normed by this induced norm, we will denote the resulting (Hilbert) space by $BL^{k+s}(\mathbb{R}^n)$. Here BL is in honour of Beppo Levi (see [2] for details), who seems to have been the first person to study these spaces. The spaces $BL^{k+s}(\mathbb{R}^n)$ then satisfy the assumptions of 2.4 with

$$\phi(r)=\begin{cases} d_{kn} r^{2k+2s-n}\ln r, & 2k+2s-n \text{ is an even integer}\\ d_{kn} r^{2k+2s-n}, & \text{otherwise}\end{cases}.$$

Here the d_{kn} are known constants whose values need not concern us. The first result is a straightforward application of 3.6, which borrows and amplifies a technique used by Powell [10] in the context of thin-plate splines.

4.1. Lemma. *Let $\{a_0, ..., a_m\}\subset\mathbb{R}^n$ be any Π_{k-1}-unisolvent set of points. Let $\alpha_0, ..., \alpha_m\in\mathbb{R}$ be chosen so that for all $p\in\Pi_{k-1}$, $\sum_{r=0}^m \alpha_r p(a_r)=0$. If $d\in\mathbb{Z}_+$ is such that $2d-2k+1\leqslant 0$, then*

$$\sum_{r,s=0}^{m} \alpha_r\alpha_s\,|a_r-a_s|^{2d}=0.$$

Proof. Using the binomial theorem,

$$\sum_{r,s=0}^{m} \alpha_r \alpha_s |a_r - a_s|^{2d}$$
$$= \sum_{r,s=0}^{m} \alpha_r \alpha_s (|a_r|^2 - 2a_r a_s + |a_s|^2)^d$$
$$= \sum_{r,s=0}^{m} \alpha_r \alpha_s \sum_{j=0}^{d} \binom{d}{j} \{|a_r|^{2(d-j)} (|a_s|^2 - 2a_r a_s)^j\}$$
$$= \sum_{r,s=0}^{m} \alpha_r \alpha_s \sum_{j=0}^{d} \binom{d}{j} \left\{|a_r|^{2(d-j)} \sum_{i=0}^{j} \binom{j}{i} |a_s|^{2(j-i)} (-2)^i (a_r a_s)^i\right\}$$
$$= \sum_{j=0}^{d} \binom{d}{j} \sum_{i=0}^{j} \binom{j}{i} (-2)^i \left\{\sum_{r=0}^{m} \alpha_r |a_r|^{2(d-j)} \sum_{s=0}^{m} \alpha_s |a_s|^{2(j-i)} (a_r a_s)^i\right\} \tag{6}$$

Now $|a_r|^{2(d-j)} (a_r a_s)^i$ can be regarded as the value of a polynomial of degree $2d-2j+i$ at a_r, while $|a_s|^{2(j-i)} (a_r a_s)^i$ can be regarded as the value at a_s of a polynomial of degree $2j-i$. Hence,

$$\sum_{r=o}^{m} \alpha_r |a_r|^{2(d-j)} a_r^i = 0 \qquad \text{if} \quad 2d-2j+i<k,$$

while

$$\sum_{s=0}^{m} \alpha_s |a_s|^{2(j-i)} a_s^i = 0 \qquad \text{if} \quad 2(j-i)+i<k.$$

Thus the whole expression in (6) is zero if for all $0 \leqslant i < j \leqslant d$, either $2d-(2j-i)-k<0$ or $2j-i-k<0$. These conditions reduce to $2j-i<k$ or $2j-i>2d-k$. All possible values will be captured if $2d-k \leqslant k-1$, that is $2d-2k+1 \leqslant 0$. ∎

4.2. THEOREM. *Let Ω be an open, connected subset of $\mathbb{R}^n$ having the cone property. For each $h>0$, let $\mathscr{A}_h$ be a finite subset of Ω with $\sup_{t \in \bar{\Omega}} \inf_{a \in \mathscr{A}_h} |t-a| \leqslant h$. Let $(n/2)-k<s<n/2$. For each $f \in BL^{k+s}(\mathbb{R}^n)$, let $U_h f$ be the minimal norm interpolant to f on $\mathscr{A}_h$. Then $|f(x)-(U_h f)(x)| = \mathcal{O}(h^{k+s-n/2})$ as $h \to 0$ for all $x \in \Omega$.*

Proof. If $2k+2s-n$ is not an even integer, then a direct application of 3.6 gives

$$|f(x)-(U_h f)(x)|^2 \leqslant c_1 d_{kn} (c_2 h)^{2k+2s-n} \langle f, f \rangle,$$

for all $f \in BL^{k+s}(\mathbb{R}^n)$. This shows that $|f(x)-(U_h f)(x)| = \mathcal{O}(h^{k+s-n/2})$ as $h \to 0$. If $2k+2s-n$ is an even integer, then a direct analysis would yield logarithmic terms in the error. Since $\phi(0)=0$, the power function has the form

$$\Phi(x) = -2 \sum_{r=1}^{\ell} p_r(x)\, \phi(|x-a_r|) + \sum_{r,s=1}^{\ell} p_r(x)\, p_s(x)\, \phi(|a_r - a_s|).$$

Set $\psi_\sigma(r) = d_{kn} r^{2k+2s-n} \ln \sigma r$, where $\sigma > 0$. Set

$$\Psi(x) = -2 \sum_{r=1}^{\ell} p_r(x)\, \psi_\sigma(|x-a_r|) + \sum_{r,s=1}^{\ell} p_r(x)\, p_s(x)\, \psi_\sigma(|a_r - a_s|).$$

Since $\psi_\sigma(r) = \phi(r) + d_{kn} r^{2k+2s-n} \ln \sigma$, we have

$$\Psi(x) = \Phi(x) - \left\{ 2 \sum_{r=1}^{\ell} p_r(x)\, |x-a_r|^{2k+2s-n} - \sum_{r,s=1}^{l} p_r(x)\, p_s(x)\, |a_r - a_s|^{2k+2s-n} \right\} d_{kn} \ln \sigma.$$

Now we intend to apply 4.1. Setting $2k+2s-n = 2d$ we see that we need $2d - 2k + 1 \leqslant 0$, so that $s \leqslant (n-1)/2$. This is precisely the condition in the theorem. Now putting $x = a_0$ and $\alpha_0 = -1$, $\alpha_r = p_r(x)$, $1 \leqslant r \leqslant \ell$,

$$\begin{aligned} 0 &= \sum_{r,s=0}^{\ell} \alpha_r \alpha_s\, |a_r - a_s|^{2k+2s-n} \\ &= \sum_{r,s=1}^{\ell} \alpha_r \alpha_s\, |a_r - a_s|^{2k+2s-n} - 2 \sum_{r=0}^{\ell} \alpha_r\, |a_0 - a_r|^{2k+2s-n} \end{aligned}$$

Thus $\Psi(x) = \Phi(x)$. Now, the analysis of 3.6 shows that

$$\begin{aligned} |f(x) - (U_h f)(x)|^2 &\leqslant c_1 \max_{0 \leqslant r \leqslant c_2 h} |\psi_\sigma(r) - \psi_\sigma(0)|\, \langle f, f \rangle \\ &\leqslant c_1 \max_{0 \leqslant r \leqslant c_2 h} |\psi_\sigma(r)|\, \langle f, f \rangle. \end{aligned}$$

Furthermore, by 3.6 c_1 and c_2 do not depend on ψ_σ, and so in particular c_1 does not depend on σ. Now,

$$\max_{0 \leqslant r \leqslant c_2 h} |\psi_\sigma(r)| = d_{kn} \max \left\{ (c_2 h)^{2k+2s-n}\, |\ln(c_2 \sigma h)|, \frac{1}{2k+2s-n} \left(\frac{1}{\sigma} \exp\left[\frac{-1}{2k+2s-n} \right] \right)^{2k+2s-n} \right\},$$

as can be seen by elementary calculus. Choosing $\sigma = 1/h$, this maximum is of order $h^{2k+2s-n}$ as required. Then $|f(x) - (U_h f)(x)| = \mathcal{O}(h^{k+s-n/2})$. ∎

It is possible to obtain improvements to the above results by assuming the function f has a higher degree of smoothness, and using a mix of techniques from Duchon [3] and the theory of finite elements (see Ciarlet [5], for example). A better version of 3.6 is what is needed. The improvements demand additional hypotheses on the domain Ω under discussion. We list these assumptions now.

4.3. Assumption. The following are useful properties of a set $\Omega \subset \mathbb{R}^n$:

(4.3.1) Ω is open, bounded and connected.

(4.3.2) Ω has the cone property.

(4.3.3) Ω has a Lipschitz boundary.

It will help to introduce the notation

$$|f|_{p,\Omega} = \left\{ \sum_{|\alpha| = k} c_\alpha \int_\Omega |(D^\alpha f)(x)|^p \, dx \right\}^{1/p},$$

where Ω is a measurable subset of $\mathbb{R}^n$, $1 \leqslant p \leqslant \infty$, and $f \in X$. Also write

$$\|f\|_{p,\Omega} = \left\{ \int_\Omega |f(x)|^p \, dx \right\}^{1/p},$$

whenever f is a pth power integrable, real-valued function on Ω. (Of course, the usual obvious modifications are to be made if $p = \infty$.) We will use $W^{k,p}(\Omega)$ to denote the usual Sobolev space of functions all of whose derivatives up to and including order k are in $L^p(\Omega)$.

4.4. Lemma (Duchon [3]). *Let $\Omega \subset \mathbb{R}^n$ satisfy* 4.3. *Let $f \in W^{k,2}(\Omega)$. Then there exists a unique element $f^\Omega \in X$ such that $f^\Omega|_\Omega = f$, and amongst all elements of X satisfying this condition, $|f^\Omega|_{2,\mathbb{R}^n}$ is minimal. Furthermore, there exists a constant $K = K(\Omega)$ such that, for all $f \in W^{k,2}(\Omega)$,*

$$|f^\Omega|_{2,\mathbb{R}^n} \leqslant K\, |f|_{2,\Omega}.$$

We need a modest strengthening of 4.4, which comes from the following change of variables result.

4.5. Lemma. *Let Ω be a measurable subset of $\mathbb{R}^n$. If the linear change of variables $\sigma(x) = t + h(x - a)$ is used, where $h > 0$, and $a, t \in \mathbb{R}^n$, then for all $u \in W^{k,p}(\Omega)$,*

$$|u|_{p,\sigma(\Omega)} = h^{n/p-k}\, |u \circ \sigma|_{p,\Omega}.$$

Proof. We have, using the change of variable formula for integration (Apostol [1]),

$$|u|^p_{p,\sigma(\Omega)} = \sum_{|\alpha|=k} c_\alpha \int_{\sigma(\Omega)} |(D^\alpha u)(x)|^p \, dx$$

$$= \sum_{|\alpha|=k} c_\alpha h^n \int_\Omega |(D^\alpha u \circ \sigma)(x)|^p \, dx.$$

Now, if $|\alpha| = k$, the

$$(D^\alpha u)(x) = [D^\alpha(u \circ \sigma \circ \sigma^{-1})](x) = [D^\alpha(u \circ \sigma)](\sigma^{-1}(x)) \, h^{-k}.$$

Hence, for such values of α,

$$(D^\alpha u \circ \sigma)(x) = (D^\alpha u)(\sigma(x)) = h^{-k}[D^\alpha(u \circ \sigma)](x).$$

Finally,

$$|u|^p_{p,\sigma(\Omega)} = \sum_{|\alpha|=k} c_\alpha h^n \int_\Omega h^{-kp} \, |[D^\alpha(u \circ \sigma)](x)|^p \, dx$$

$$= h^{n-kp} \, |u \circ \sigma|^p_{p,\Omega}. \quad \blacksquare$$

4.6. Lemma. *Let B be any ball of radius h in $\mathbb{R}^n$. Let $f \in W^{k,2}(B)$. Then there exists a unique element $f^B \in X$ such that $f^B|_B = f$ and amongst all such elements of X, $|f^B|_{2,\mathbb{R}^n}$ is minimal. Moreover, there exists a constant C, independent of B, such that for all $f \in W^{k,2}(B)$,*

$$|f^B|_{2,\mathbb{R}^n} \leqslant C \, |f|_{2,B}.$$

Proof. This result is identical to 4.4 except for the fact that C can be taken independent of B. To see this, let $B = \{x \in \mathbb{R}^n \colon |x - a| \leqslant h\}$ and define $\sigma(x) = h^{-1}(x - a)$. Let B_0 denote the set $\{x \in \mathbb{R}^n \colon |x| \leqslant 1\}$. Then $\sigma(B) = B_0$. Take $f \in W^{k,2}(B)$. Then $f \circ \sigma^{-1} \in W^{k,2}(B_0)$. Set $F = f \circ \sigma^{-1}$. It is an elementary property of the semi-norm that $F^{B_0} = f^B \circ \sigma^{-1}$. By 4.4, $|f^B \circ \sigma^{-1}|_{2,\mathbb{R}^n} \leqslant K(B_0) \, |f^B \circ \sigma^{-1}|_{2,B_0}$. Using 4.5, we obtain $h^{n/2} \, |f^B|_{2,\mathbb{R}^n} \leqslant K(B_0) \, h^{n/2} \, |f^B|_{2,B}$. Taking $C = K(B_0)$ concludes the proof. $\blacksquare$

4.7. Theorem (Duchon). *Let Ω be a subset of $\mathbb{R}^n$ satisfying* 4.3, *let $1 \leqslant p \leqslant \infty$ and let $2k > n$. For each $h > 0$, let $\mathscr{A}_h$ be a finite, Π_{k-1}-unisolvent subset of Ω with $\sup_{t \in \bar{\Omega}} \inf_{a \in \mathscr{A}_h} |t - a| \leqslant h$. For each $f \in W^{k,2}(\Omega)$ let $U_h f$ be the minimal norm interpolant to f on $\mathscr{A}_h$, so that $U_h f \in BL^k(\mathbb{R}^n)$. There* *exists a constant $h_0 > 0$ and a constant $C > 0$, independent of h, such that for all $f \in W^{k,2}(\Omega) \cap L^p(\Omega)$,*

$$\|f - U_h f\|_{p,\Omega} \leqslant \begin{cases} Ch^{k-(n/2)+(n/p)}\, |f|_{2,\Omega} & 2 \leqslant p \leqslant \infty \\ Ch^k\, |f|_{2,\Omega} & 1 \leqslant p < 2 \end{cases}.$$

Proof. We begin as in 3.6 and follow that proof until $a_1, ..., a_\ell$ have been constructed in $B(t, MRh)$. We do not apply 2.8, however. Instead we will firstly define f^Ω in accordance with 4.4. For a large part of the proof we wish to work with f^Ω and not f. For convenience, we will henceforth write f for f^Ω, and U for U_h. Now use B to denote $B(t, MRh)$ and define $(f - Uf)^B$ in accordance with 4.4, so that $(f - Uf)^B|_B = (f - Uf)|_B$. Then $(f - Uf)^B(a_i) = 0$, $i = 1, ..., \ell$, and so 2.9 can be used to give

$$\{f(x) - (Uf)(x)\}^2 = \{(f - Uf)^B(x)\}^2 \leqslant \Phi(x)\, |(f - Uf)^B|_{2,\mathbb{R}^n}.$$

Now, using 4.6 gives

$$\{f(x) - (Uf)(x)\}^2 \leqslant C\Phi(x)\, |f - Uf|^2_{2,B},$$

where C is independent of the choice of $B(t, MRh)$. Thus

$$\|f - Uf\|_{p,B} \leqslant \sqrt{C}\, |f - Uf|_{2,B} \left\{ \int_B |\Phi(x)|^{p/2}\, dx \right\}^{1/p}. \qquad (7)$$

Now argue as in 3.6 again to realise Φ as

$$\Phi(x) = -2 \sum_{r=1}^{\ell} p_r(x)[\phi(|x - a_r|) - \phi(0)] + \sum_{r,s=1}^{\ell} p_r(x)\, p_s(x)[\phi(|a_r - a_s|) - \phi(0)].$$

Then, since $\phi(0) = 0$,

$$\begin{aligned} \int_B |\Phi(x)|^{p/2}\, dx &= \int_B \left| -2 \sum_{r=1}^{\ell} p_r(x)\, \phi(|x - a_r|) + \sum_{r,s=1}^{\ell} p_r(x)\, p_s(x)\, \phi(|a_r - a_s|) \right|^{p/2} dx \\ &\leqslant \int_B \left\{ 2 \sum_{r=1}^{\ell} |p_r(x)\, \phi(|x - a_r|)| + \sum_{r,s=1}^{\ell} |p_r(x)\, p_s(x)\, \phi(|a_r - a_s|)| \right\}^{p/2} dx \end{aligned}$$

Define $\lambda = \max\{\phi(r): 0 \leqslant r \leqslant MRh\}$, and $\Lambda = \max\{\sum_{r=1}^{\ell} |p_r(x)|: x \in B(t, MRh)\}$. Then

$$\int_B |\Phi(x)|^{p/2}\, dx \leqslant \lambda^{p/2} \int_B \left\{2 \sum_{r=1}^{\ell} |p_r(x)| + \sum_{r,\, s=1}^{\ell} |p_r(x) p_s(x)|\right\}^{p/2} dx$$
$$\leqslant \lambda^{p/2} \int_B (2\Lambda + \Lambda^2)^{p/2}\, dx$$
$$\leqslant K_1 \lambda^{p/2} h^n (2\Lambda + \Lambda^2)^{p/2}.$$

Now, from (7), setting $K_2^p = K_1 C^{p/2} (2\Lambda + \Lambda^2)^{p/2}$,

$$\|f - Uf\|_{p,\, B} \leqslant K_2 \lambda^{1/2} h^{n/p}\, |f - Uf|_{2,\, B}. \tag{8}$$

Set $\Omega^* = \bigcup_{t \in T_h} B(t, MRh)$. Using the fact that if $a \in \mathbb{R}^m$ than $\|a\|_p \leqslant \|a\|_2$ for $p \geqslant 2$, we have

$$\|f - Uf\|_{p,\, \Omega} \leqslant \|f - Uf\|_{p,\, \Omega^*}$$
$$\leqslant \left\{\sum_{t \in T_h} \|f - Uf\|_{p,\, B(t,\, MRh)}^p\right\}^{1/p}$$
$$\leqslant K_2 \lambda^{1/2} h^{n/p} \left\{\sum_{t \in T_h} |f - Uf|_{2,\, B(t,\, MRh)}^p\right\}^{1/p}$$
$$\leqslant K_2 \lambda^{1/2} h^{n/p} \left\{\sum_{t \in T_h} |f - Uf|_{2,\, B(t,\, MRh)}^2\right\}^{1/2}$$
$$\leqslant K_2 \lambda^{1/2} h^{n/p} \left\{\sum_{t \in T_h} \int_{\mathbb{R}^n} \chi_{B(t,\, MRh)} \left(\sum_{|\alpha| = k} c_\alpha |D^\alpha (f - Uf)|^2\right)\right\}^{1/2}$$
$$\leqslant K_2 \lambda^{1/2} h^{n/p} \left\{\int_{\mathbb{R}^n} \left(\sum_{|\alpha| = k} c_\alpha |D^\alpha (f - Uf)|^2\right) \sum_{t \in T_h} \chi_{B(t,\, MRh)}\right\}^{1/2}.$$

Now, using 3.5(iii),

$$\|f - Uf\|_{p,\, \Omega} \leqslant K_2 (M_1)^{1/2} \lambda^{1/2} h^{n/p}\, |f - Uf|_{2,\, \mathbb{R}^n}.$$

At this stage, the reader should recall that f is being used to denote f^Ω, and U is being used to denote U_h. Thus, the above inequality may be rewritten by asserting that there exist constants K_3 and K_4 such that

$$\|f - U_h f\|_{p,\, \Omega} \leqslant K_3 \lambda^{1/2} h^{n/p}\, |f^\Omega - U_h f^\Omega|_{2,\, \mathbb{R}^n}$$
$$\leqslant K_3 \lambda^{1/2} h^{n/p}\, |f^\Omega|_{2,\, \mathbb{R}^n}$$
$$\leqslant K_4 \lambda^{1/2} h^{n/p}\, |f|_{2,\, \Omega}.$$

Finally, the proof of 4.2 shows that there is a constant K_5 such that $\lambda^{1/2} \leqslant K_5 h^{k-n/2}$.

One needs slightly different techniques to handle the case where $1 \leqslant p < 2$. Following the arguments of the proof as far as Eq. (7), we have

$$\|f - Uf\|_{p,B} \leqslant K_2 \lambda^{1/2} h^{n/p} |f - Uf|_{2,B}.$$

Now using Holder's inequality with $(p/2) + (1/q) = 1$, which is valid for $1 \leqslant p \leqslant 2$,

$$\begin{aligned}
\|f - Uf\|_{p,\Omega} &\leqslant \|f - Uf\|_{p,\Omega^*} \\
&\leqslant \left\{ \sum_{t \in T_h} \|f - Uf\|^p_{p, B(t, MRh)} \right\}^{1/p} \\
&\leqslant K_2 \lambda^{1/2} h^{n/p} \left\{ \sum_{t \in T_h} |f - Uf|^p_{2, B(t, MRh)} \right\}^{1/p} \\
&\leqslant K_2 \lambda^{1/2} h^{n/p} \left\{ \sum_{t \in T_h} |f - Uf|^2_{2, B(t, MRh)} \right\}^{1/2} \left\{ \sum_{t \in T_h} 1^q \right\}^{1/pq} \\
&\leqslant K_2' \lambda^{1/2} h^{n/p} \left\{ \sum_{t \in T_h} |f - Uf|^2_{2, B(t, MRh)} \right\}^{1/2} h^{-n/pq},
\end{aligned}$$

where K_2' is a suitable constant. Now using 3.5(iii),

$$\|f - Uf\|_{p,\Omega} \leqslant K_2' M_1^{1/2} \lambda^{1/2} h^{n/2} |f - Uf|_{2,\mathbb{R}^n}.$$

The proof now continues as before, leading to the conclusion that $\|f - Uf\|_p \leqslant K_5 h^k |f|_{2,\Omega}$. ∎

Finally, we have chosen to illustrate the sort of analysis needed by looking at the surface splines of Duchon, and credited the final result to Duchon. Strictly speaking, the result in [3] concerns the case $2 \leqslant p \leqslant \infty$, but we do not regard the case $1 \leqslant p < 2$ as differing in a substantial way from the previous case, although the form of the error bound is quite interesting. Our analysis in this section is also helped by the fact that tools are available from the Sobolev theory. A similar analysis for radial functions which are conditionally positive definite of some particular order might be somewhat more delicate.

ACKNOWLEDGMENTS

We acknowledge with gratitude the helpful comments on the first draft of this paper by Professor R. Schaback and the careful reading of the referees.

REFERENCES

1. T. Apostol, "Mathematical Analysis," 2nd ed., 1974.
2. J. Deny and J. L. Lions, Les espaces du type de Beppo Levi, *Ann. Inst. Fourier* **5** (1954), 305–370.
3. J. Duchon, Sur l'erreur d'interpolation des fonctions de plusieurs variables par les D^m-splines, *RAIRO Anal. Numer.* **12** No. 4 (1978), 325–334.
4. J. Duchon, Splines minimising rotation-invariant seminorms in Sobolev spaces, *in* "Constructive theory of Functions of Several Variables" (W. Schempp and K. Zeller, Eds.), Lecture Notes in Mathematics, Vol. 571, pp. 85–100, Springer-Verlag, Berlin, 1977.
5. P. G. Ciarlet, "The finite element method for elliptic problems," North-Holland, Amsterdam, 1978.
6. M. Golomb and H. F. Weinberger, Optimal approximation and error bounds, *in* "On Numerical Approximation" (R. E. Langer, Ed.), pp. 117–190, University of Wisconsin Press, Madison, 1959.
7. W. R. Madych and S. A. Nelson, Multivariate interpolation and conditionally positive definite functions, II, *Math. Comp.* **54**, 211–230.
8. W. Madych and S. Nelson, Multivariate interpolation and conditionally positive definite functions, *J. Approx. Theory Appl.* **4**, No. 4 (1988), 77–89.
9. J. Meinguet, Multivariate interpolation at arbitrary points made simple, *Z. Angew. Math. Phys.* **30**, 292–304.
10. M. J. D. Powell, The uniform convergence of thin plate spline interpolation in two dimensions, *Numer. Math.* **68**, No. 1 (1994), 107–128.
11. W. Rudin, "Functional Analysis," McGraw–Hill, New York, 1973.
12. R. Schaback, A comparison of radial basis function interpolants, *in* "Multivariate approximation—from CAGD to wavelets" (K. Jetter and F. Utreras, Eds.), pp. 293–305, World Scientific, London, 1993.